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Hermann Weyl's Space-Time Geometry and the Origin of Gauge Theory 100 Years ago

Norbert Straumann, Physik-Institut, Uni Zürich

One of the major developments of twentieth century physics has been the gradual recognition that a common feature of the known fundamental interactions is their gauge structure. In this contribution the early history of gauge theory is reviewed, emphasizing mainly Hermann Weyl's seminal contributions of 1918 and 1929. Wolfgang Pauli's early construction in 1953 of a non-Abelian Kaluza-Klein theory is described in some detail.

1 Introduction

The history of gauge theories begins with General Relativity, which can be regarded as a non-Abelian gauge theory of a special type. To a large extent the other gauge theories emerged in a slow and complicated process gradually from General Relativity. Their common geometrical structure – best expressed in terms of connections of fiber bundles – is now widely recognized.

Figure 1: Hermann Weyl.

It all began with H. Weyl [1], who made in 1918 the first attempt to extend General Relativity in order to describe gravitation and electromagnetism within a unifying geometrical framework. This brilliant proposal contains the germs of all *mathematical* aspects of non-Abelian gauge theory. The word 'gauge' (german: 'Eich-') transformation appeared for the first time in this paper, but in the everyday meaning of change of length or change of calibration.

Einstein admired Weyl's theory as "*a coup of genius of the first rate*", but immediately realized that it was physically untenable. After a long discussion Weyl finally admitted that his attempt was a failure as a physical theory. (For a discussion of the intense Einstein-Weyl correspondence, see Ref. [2].) It paved, however, the way for the correct understanding of gauge invariance. Weyl himself reinterpreted in 1929 his original theory after the advent of quantum theory in a grand paper [3]. Weyl's reinterpretation of his earlier speculative proposal had actually been suggested before by London [12]. Fock [16], Klein [35], and others arrived at the principle of gauge invariance in the framework of wave mechanics along a completely different line. It was, however, Weyl who emphasized the role of gauge invariance as a *constructive principle* from which electromagnetism can be derived. This point of view became very fruitful for our present understanding of fundamental interactions. (For a more extensive discussion, see [4].)

2 Weyl's Attempt to Unify Gravitation and Electromagnetism

On the 1st of March 1918 Weyl writes in a letter to Einstein ([5], Vol. 8B, Doc.472): "*These days I succeeded, as I believe, to derive electricity and gravitation from a common source ...*". Einstein's prompt reaction by postcard indicates already a physical objection which he explained in detail shortly afterwards. Before we come to this we have to describe Weyl's theory of 1918.

Weyl's starting point was purely mathematical. He felt a certain uneasiness about Riemannian geometry, as is clearly expressed by the following sentences early in his paper:

> *But in Riemannian geometry described above there is contained a last element of geometry "at a distance" (ferngeometrisches Element) — with no good reason, as far as I can see; it is due only to the accidental development of Riemannian geometry from Euclidean geometry. The metric allows the two magnitudes of two vectors to be compared, not only at the same point, but at any arbitrarily separated points. A true infinitesimal geometry should, however, recognize only a principle for transferring the magnitude of a vector to an infinitesimally close point and then, on transfer to an arbitrary distant point, the integrability of the magnitude of a vector is no more to be expected than the integrability of its direction.*

After these remarks Weyl turns to physical speculation and continues as follows:

> *On the removal of this inconsistency there appears a geometry that, surprisingly, when applied to the world, explains not only the gravitational phenomena but also the electrical. According to the resultant theory both spring from the same source, indeed in general one cannot separate gravitation and electromagnetism in a unique manner. In this theory all physical quantities have a world geometrical meaning; the action appears from the beginning as a pure number. It leads to an essentially unique universal law; it even allows us to understand in a certain sense why the world is four-dimensional.*

3 Weyl's Generalization of Riemannian Geometry

In this section we describe in some detail Weyl's geometry in a bundle theoretical language. I prefer this, because it is common with that of non-abelian gauge theories (on the classical level).

In Weyl's geometry the spacetime manifold M is equipped with a *conformal structure*, i.e., with a class $[g]$ of conformally equivalent Lorentz metrics g (and not a definite metric as in General Relativity). For such a conformal manifold

(M, [g]) we can introduce the bundle of *conformal frames*, which are linear frames (X_0,X_1,X_2,X_3) for which $g_p(X_\mu,X_\nu) = \exp(2\lambda(p))\eta_{\mu\nu}$, where $\eta = (\eta_{\mu\nu}) = \text{diag}\,(-1,1,1,1)$, for any (and thus all) $g \in [g]$. The set $W(M)$ of conformal frames on M can be regarded in an obvious manner as the total space of a principle fibre bundle, whose structure group G is the group consisting of all positive multiples of homogeneous Lorentz transformations, i.e., $G \cong O(1,3) \times \mathbb{R}_+$. This *conformal (Weyl) bundle* is a reduction of the bundle of linear frames $L(M)$ (and an extension of the bundle of orthonormal frames for every $g \in [g]$). A *Weyl connection* is a torsion-free connection on $W(M)$, defined by a connection form ω. (As such it has a unique extension to $L(M)$.) The canonical 1-form θ on $W(M)$, i.e., the restriction of the soldering form on $L(M)$, satisfies $D^\omega\theta = 0$, where D^ω is the exterior covariant derivative belonging to ω, expressing the vanishing torsion. Since the connection form has values in the Lie algebra $\mathcal{G}$ of G, i.e., in $o(1,3) \oplus \mathbb{R}$, we can split ω uniquely

$$\omega = \hat{\omega} + \phi \cdot 1, \qquad (1)$$

where $\hat{\omega}$ has values in $o(1,3)$ and ϕ is an $\mathbb{R}$-valued 1-form on $W(M)$. Thus in matrix notation

$$\hat{\omega}^T\eta + \eta\hat{\omega} = 0, \quad \omega^T\eta + \eta\omega = 2\phi\eta. \qquad (2)$$

A Weyl connection can be considered as a torsion free linear connection, which is reducible to a connection in $W(M)$. The restriction of $\hat{\omega}$ to any orthonormal frame bundle $O_g(M) \subset W(M)$, $g \in [g]$, defines a metric connection in $O_g(M)$ with torsion. Since the torsion of the Weyl connection vanishes, the first structure equation reads

$$d\theta + \omega \wedge \theta = 0. \qquad (3)$$

The curvature $\Omega = D^\omega\omega$ is determined by the second structure equation

$$\Omega = d\omega + \omega \wedge \omega, \qquad (4)$$

which can be written as

$$\Omega = (d\hat{\omega} + \hat{\omega} \wedge \hat{\omega}) + d\phi \cdot 1. \qquad (5)$$

A *Weyl space* is a conformal manifold together with a Weyl connection.
The frames $\sigma(x) = \{e_\mu(x)\}$ of a local section $\sigma : U \to W(M)$ are dual to to the components θ^μ of $\sigma^*\theta$,

$$\theta^\mu(e_\nu) = \delta^\mu_\nu. \qquad (6)$$

For any metric $g \in [g]$ we can choose local sections such that the frames $\{e_\mu(x)\}$ are orthonormal with respect to g,

$$g = \eta_{\mu\nu}\theta^\mu \otimes \theta^\nu. \qquad (7)$$

The exterior covariant derivative of g has relative to the dual basis $\{\theta^\mu\}$ the components [1]

$$(Dg)_{\mu\nu} = d\eta_{\mu\nu} - \omega^\lambda_\mu\eta_{\lambda\nu} - \omega^\lambda_\nu\eta_{\lambda\mu} \overset{(2)}{=} -2A\eta_{\mu\nu}, \qquad (8)$$

with $A := \sigma^*\phi$. Thus

$$Dg = -2A \otimes g. \qquad (9)$$

If g is replaced by $\tilde{g} = e^{2\lambda}g \in [g]$ then $D\tilde{g} = -2\tilde{A} \otimes \tilde{g}$, where $\tilde{A} = A - d\lambda$.

This leads us to the concept of a covariant Weyl derivative on a conformal manifold (M, [g]): A *covariant Weyl derivative* ∇ on a conformal manifold (M, [g]) is a covariant torsionless derivative on the spacetime manifold M which satisfies the condition

$$\nabla g = -2A \otimes g, \qquad (10)$$

where the map $A : [g] \to \Lambda^1(M)$ satisfies

$$A(e^{2\lambda}g) = A(g) - d\lambda. \qquad (11)$$

$A(g)$ is the *gauge potential* belonging to g, and (11) is what Weyl called a *gauge transformation*.

It is not difficult to show that there is a bijective relation between the set of covariant Weyl derivatives on a conformal manifold (M, [g]) and the set of Weyl connection forms on the corresponding conformal bundle.

Existence of covariant Weyl derivatives

For the existence and explicit formulae of covariant Weyl derivatives we generalize the well-known Koszul treatment of the Levi-Civita connection. In particular we generalize the Koszul formula (see, e.g., [21], eq. (15.42)) for the covariant Levi-Civita derivative ∇^{LC} to

$$g(\nabla_Z Y, X) = g(\nabla^{LC}_Z Y, X) + [-A(X)g(Y,Z) + A(Y)g(Z,X) + A(Z)g(X,Y)]. \qquad (12)$$

This equation defines ∇_X in terms of g and A.
Derivation of (12). Equation (10) reads explicitly

$$(\nabla_X g)(Y,Z) = Xg(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z) = -2A(X)g(Y,Z). \qquad (13)$$

Since the torsion vanishes, i.e., $\nabla_X Y - \nabla_Y X - [X,Y] = 0$, we can write this as

$$Xg(Y,Z) = g(\nabla_Y X, Z) + g([X,Y],Z) + g(Y,\nabla_X Z) + 2A(X)g(Y,Z). \qquad (14)$$

After cyclic permutations, we obtain as in the derivation of the standard Koszul formula, equation (12).

With routine calculations one verifies that the generalized Koszul formula (12) defines a covariant derivative with vanishing torsion, and moreover it satisfies the defining property (10). (In these calculations one uses that the Levi-Civita derivative has vanishing torsion and that the metricity of ∇^{LC} is equivalent to the Ricci identity [21], eq. (15.39)).

Local formula. Choose in (12) $X = \partial_\mu$, $Y = \partial_\nu$, $Z = \partial_\lambda$ of local coordinates. Then we obtain

$$\langle \nabla_{\partial_\mu}\partial_\nu, \partial_\lambda \rangle = \frac{1}{2}(-g_{\nu\mu,\lambda} + g_{\mu\lambda,\nu} + g_{\lambda\nu,\mu}) + (-A_\mu g_{jk} + A_\nu g_{\nu\lambda} + A_\lambda g_{\mu\nu}). \qquad (15)$$

1 In the local equations ω^α_β denotes the pull-back $\sigma^*\omega^\alpha_\beta$.

In other words, one has to perform in the Christoffel symbols of the Levi-Civita connection the substitution

$$g_{\mu\nu,\lambda} \rightarrow g_{\mu\nu,\lambda} - 2A_\lambda g_{\mu\nu}. \qquad (16)$$

Consider now a curve $\gamma : [0, 1] \rightarrow M$ and a parallel-transported vector field X along γ. If $l(t)$ is the length of $X(t)$, measured with a representative $g \in [g]$, we obtain from (10)

$$\frac{\dot{l}}{l} = \frac{1}{2l^2}(\nabla_{\dot{\gamma}} g)(X(t), X(t)) = -A(\dot{\gamma}), \qquad (17)$$

and thus the following relation between $l(p)$ for the initial point $p = \gamma(0)$ and $l(q)$ for the end point $q = \gamma(1)$:

$$l(q) = \exp\left(-\int_\gamma A\right) l(p). \qquad (18)$$

Equation (11) implies that this relation holds for all $g \in [g]$. Therefore, the ratio of lengths in q and p (measured with $g \in [g]$) *depends in general on the connecting path* γ (see Fig. 2). The length is only independent of γ if the exterior differential of A,

$$F = dA \qquad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu), \qquad (19)$$

vanishes.
Note that (18) holds in particular for a geodesic $(\nabla_{\dot{\gamma}}\dot{\gamma} = 0)$ and $X = \dot{\gamma}$. So the length ot the tangent vector $\dot{\gamma}$ does not remain constant as in the pseudo-Riemannian case.

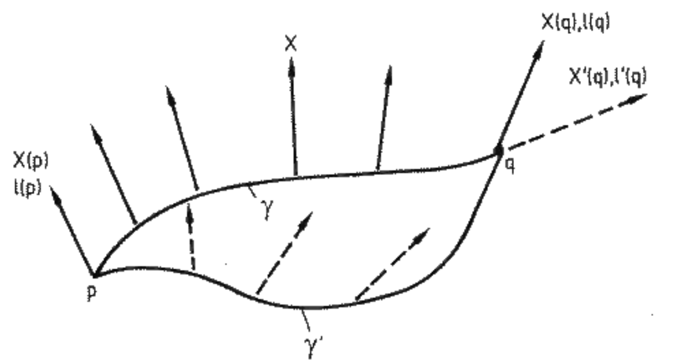

Figure 2: Path dependence of parallel displacement and transport of length in Weyl spacetime.

4 Electromagnetism and Gravitation

Turning to physics, Weyl assumes that his "purely infinitesimal geometry" describes the structure of spacetime and consequently he requires that physical laws should satisfy a double-invariance:
1. They must be invariant with respect to arbitrary smooth coordinate transformations.
2. They must be *gauge invariant*, i.e., invariant with respect to substitutions

$$g \rightarrow e^{2\lambda} g, \qquad A \rightarrow A - d\lambda, \qquad (20)$$

for an arbitrary smooth function λ.

Nothing is more natural to Weyl, than identifying A_μ with the vector potential and $F_{\mu\nu}$ in eq. (19) with the field strength of electromagnetism. In the absence of of electromagnetic fields ($F_{\mu\nu} = 0$) the scale factor $\exp(-\int_\gamma A)$ in (18) for length transport becomes path independent (integrable) and one can find a gauge such that A_μ vanishes for simply connected spacetime regions. In this special case one is in the same situation as in General Relativity.

Weyl proceeds to find an action which is generally invariant as well as gauge invariant and which would give the coupled field equations for g and A. We do not want to enter into this, except for the following remark. In his first paper [1] Weyl proposes what we call nowadays the Yang-Mills action

$$S(g, A) = -\frac{1}{4}\int Tr(\Omega \wedge *\Omega). \qquad (21)$$

Here Ω denotes the curvature form and $*\Omega$ its Hodge dual. Note that the latter is gauge invariant, i.e., independent of the choice of $g \in [g]$. In Weyl's geometry the curvature form splits as $\Omega = \hat{\Omega} + F$, where $\hat{\Omega}$ is the metric piece [9]. Correspondingly, the action also splits,

$$S(g, A) = -\frac{1}{4}\int Tr(\hat{\Omega} \wedge *\hat{\Omega}) - \frac{1}{4}\int F \wedge *F. \qquad (22)$$

The second term is just the Maxwell action. Weyl's theory thus contains formally all aspects of a non-Abelian gauge theory [2].

Weyl emphasizes, of course, that the Einstein-Hilbert action is not gauge invariant. Later work by Pauli [10] and by Weyl himself [7, 1] led soon to the conclusion that the action (21) could not be the correct one, and other possibilities were investigated (see the later editions of Weyl's classic treatise [7]).

Independent of the precise form of the action Weyl shows that in his theory gauge invariance implies the *conservation of electric charge* in much the same way as general coordinate invariance leads to the conservation of energy and momentum [3]. This beautiful connection pleased him particularly: "*... [it] seems to me to be the strongest general argument in favour of the present theory — insofar as it is permissible to talk of justification in the context of pure speculation.*" The invariance principles imply five 'Bianchi type' identities. Correspondingly, the five conservation laws follow in two independent ways from the coupled field equations and may be "termed the eliminants" of the latter. These structural connections hold also in modern gauge theories.

4.1 Einstein's Objection and Reactions of Other Physicists

After this sketch of Weyl's theory we come to Einstein's striking counterargument which he first communicated to Weyl by postcard. The problem is that if the idea of a nonintegrable length connection (scale factor) is correct, then the behavior of clocks would depend on their history. Consider two identical atomic clocks in adjacent world points and bring them along different world trajectories which meet again in adjacent world points. According to (21) their frequencies would then generally differ. This is in clear contradiction with empirical evidence, in particular with the existence of stable

2 The integrand in equation (21) is in local coordinates indeed identical to the scalar density $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\sqrt{-g}\,dx^0 \wedge \ldots \wedge dx^3$ which is used by Weyl ($R_{\alpha\beta\gamma\delta}$= the curvature tensor of the Weyl connection).

3 We adopt here the somewhat naive interpretation of energy-momentum conservation for generally invariant theories of the older literature.

atomic spectra. Einstein therefore concludes (see [5], Vol. 8B, Doc. 507):

... (if) one drops the connection of the ds to the measurement of distance and time, then relativity looses all its empirical basis.

Nernst shared Einstein's objection and demanded on behalf of the Berlin Academy that it should be printed in a short amendment to Weyl's article. Weyl had to accept this. We have described the intense and instructive subsequent correspondence between Weyl and Einstein elsewhere [2] (see also Vol. 8B of [5]). As an example, let us quote from one of the last letters of Weyl to Einstein ([5], Vol. 8B, Doc. 669):

This [insistence] irritates me of course, because experience has proven that one can rely on your intuition; so unconvincing as your counterarguments seem to me, as I have to admit ...

By the way, you should not believe that I was driven to introduce the linear differential form in addition to the quadratic one by physical reasons. I wanted, just to the contrary, to get rid of this 'methodological inconsistency (Inkonsequenz)' which has been a bone of contention to me already much earlier. And then, to my surprise, I realized that it looked as if it might explain electricity. You clap your hands above your head and shout: But physics is not made this way ! (Weyl to Einstein 10.12.1918).

Weyl's reply to Einstein's criticism was, generally speaking, this: The real behavior of measuring rods and clocks (atoms and atomic systems) in arbitrary electromagnetic and gravitational fields can be deduced only from a dynamical theory of matter.

Not all leading physicists reacted negatively. Einstein transmitted a very positive first reaction by Planck, and Sommerfeld wrote enthusiastically to Weyl that there was *"... hardly doubt, that you are on the correct path and not on the wrong one."*

In his encyclopedia article on relativity [11] Pauli gave a lucid and precise presentation of Weyl's theory, but commented on Weyl's point of view very critically. At the end he states:

... In summary one may say that Weyl's theory has not yet contributed to getting closer to the solution of the problem of matter.

Also Eddington's reaction was at first very positive but he soon changed his mind and denied the physical relevance of Weyl's geometry.

The situation was later appropriately summarized by F. London in his 1927 paper [12] as follows:

In the face of such elementary experimental evidence, it must have been an unusually strong metaphysical conviction that prevented Weyl from abandoning the idea that Nature would have to make use of the beautiful geometrical possibility that was offered. He stuck to his conviction and evaded discussion of the above-mentioned contradictions through a rather unclear re-interpretation of the concept of "real state", which, however, robbed his theory of its immediate physical meaning and attraction.

In this remarkable paper, London suggested a reinterpretation of Weyl's principle of gauge invariance within the new quantum mechanics: The role of the metric is taken over by the wave function, and the rescaling of the metric has to be replaced by a phase change of the wave function.

In this context an astonishing early paper by Schrödinger [13] has to be mentioned, which also used Weyl's "World Geometry" and is related to Schrödinger's later invention of wave mechanics. This relation was discovered by V. Raman and P. Forman [14]. (See also the discussion by C. N. Yang in [15].)

Even earlier than London, V. Fock [16] arrived along a completely different line at the principle of gauge invariance in the framework of wave mechanics. His approach was similar to the one by O. Klein [35].

The contributions by Schrödinger [13], London [12] and Fock [16] are commented in [8], where also English translations of the original papers can be found. Here, we concentrate on Weyl's seminal paper *Electron and Gravitation*.

5 Weyl's 1929 Classic: *Electron and Gravitation*

Shortly before his death late in 1955, Weyl wrote for his *Selecta* [17] a postscript to his early attempt in 1918 to construct a 'unified field theory'. There he expressed his deep attachment to the gauge idea and adds (p. 192):

Later the quantum-theory introduced the Schrödinger-Dirac potential ψ of the electron-positron field; it carried with it an experimentally-based principle of gauge-invariance which guaranteed the conservation of charge, and connected the ψ with the electromagnetic potentials A_μ in the same way that my speculative theory had connected the gravitational potentials $g_{\mu\nu}$ with the A_μ, and measured the A_μ in known atomic, rather than unknown cosmological units. I have no doubt but that the correct context for the principle of gauge-invariance is here and not, as I believed in 1918, in the intertwining of electromagnetism and gravity.

This re-interpretation was developed by Weyl in one of the great papers of the 20th century [3]. Weyl's classic does not only give a very clear formulation of the gauge principle, but contains, in addition, several other important concepts and results — in particular his two-component spinor theory.

The modern version of the gauge principle is already spelled out in the introduction:

The Dirac field-equations for ψ together with the Maxwell equations for the four potentials f_p of the electromagnetic field have an invariance property which is formally similar to the one which I called gauge-invariance in my 1918 theory of gravitation and electromagnetism; the equations remain invariant when one makes the simultaneous substitutions

$$\psi \text{ by } e^{i\lambda}\psi \quad \text{and} \quad f_p \text{ by } f_p - \frac{\partial\lambda}{\partial x^p},$$

where λ is understood to be an arbitrary function of position in four-space. Here the factor e/ch, where –e is the charge of the electron, c is the speed of light, and h/2π is the quantum of action, has been absorbed in f_p. The connection of this "gauge invariance" to the conservation of electric charge remains untouched. But a fundamental difference, which is important to obtain agreement with observation, is that the exponent of the factor multiplying ψ is not real but pure imaginary. ψ now plays the role that Einstein's ds played before. It seems to me that this new principle of gauge-invariance, which follows not from speculation but from experiment, tells us that the electromagnetic field is a necessary accompanying phenomenon, not of gravitation, but of the material wave-field represented by ψ. Since gauge-invariance involves an arbitrary function λ it has the character of "general" relativity and can naturally only be understood in that context.

We shall soon enter into Weyl's justification which is, not surprisingly, strongly associated with General Relativity. Before this we have to describe his incorporation of the Dirac theory into General Relativity which he achieved with the help of the tetrad formalism.

One of the reasons for adapting the Dirac theory of the spinning electron to gravitation had to do with Einstein's recent unified theory which invoked a distant parallelism with torsion. E. Wigner [18] and others had noticed a connection between this theory and the spin theory of the electron. Weyl did not like this and wanted to dispense with teleparallelism. In the introduction he says:

I prefer not to believe in distant parallelism for a number of reasons. First my mathematical intuition objects to accepting such an artificial geometry; I find it difficult to understand the force that would keep the local tetrads at different points and in rotated positions in a rigid relationship. There are, I believe, two important physical reasons as well. The loosening of the rigid relationship between the tetrads at different points converts the gauge-factor $e^{i\lambda}$, which remains arbitrary with respect to ψ, from a constant to an arbitrary function of space-time. In other words, only through the loosening the rigidity does the established gauge-invariance become understandable.

This thought is carried out in detail after Weyl has set up his two-component theory in special relativity, including a discussion of *P* and *T* invariance. He emphasizes thereby that the two-component theory excludes a linear implementation of parity and remarks: "*It is only the fact that the left-right symmetry actually appears in Nature that forces us to introduce a second pair of ψ-components.*" To Weyl the mass-problem is thus not relevant for this [4]. Indeed he says: "*Mass, however, is a gravitational effect; thus there is hope of finding a substitute in the theory of gravitation that would produce the required corrections.*"

4 At the time it was thought by Weyl, and indeed by all physicists, that the 2-component theory requires a zero mass. In 1957, after the discovery of parity nonconservation, it was found that the 2-component theory could be consistent with a finite mass. See K. M. Case, [19].

5.1 Tetrad Formalism

In order to incorporate his two-component spinors into General Relativity, Weyl was forced to make use of local tetrads (Vierbeine). In section 2 of his paper he develops the tetrad formalism in a systematic manner. This was presumably independent work, since he does not give any reference to other authors. It was, however, mainly E. Cartan who demonstrated with his work [20] the usefulness of locally defined orthonormal bases – also called moving frames – for the study of Riemannian geometry.

In the tetrad formalism the metric is described by an arbitrary basis of orthonormal vector fields $\{e_\alpha(x);\, \alpha = 0, 1, 2, 3\}$. If $\{e^\alpha(x)\}$ denotes the dual basis of 1-forms, the metric is given by

$$g = \eta_{\mu\nu}\, e^\mu(x) \otimes e^\nu(x), \quad (\eta_{\mu\nu}) = diag(1, -1, -1, -1). \qquad (23)$$

Weyl emphasizes, of course, that only a class of such local tetrads is determined by the metric: the metric is not changed if the tetrad fields are subject to spacetime-dependent Lorentz transformations:

$$e^\alpha(x) \to \Lambda^\alpha_\beta(x)\, e^\beta(x). \qquad (24)$$

With respect to a tetrad, the connection forms $\omega = (\omega^\alpha_\beta)$ have values in the Lie algebra of the homogeneous Lorentz group:

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0. \qquad (25)$$

(Indices are raised and lowered with $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$, respectively.) They are determined (in terms of the tetrad) by the first structure equation of Cartan:

$$de^\alpha + \omega^\alpha_\beta \wedge e^\beta = 0. \qquad (26)$$

(For a textbook derivation see, e.g., [21], especially Sects. 2.6 and 8.5.) Under local Lorentz transformations (24) the connection forms transform in the same way as the gauge potential of a non-Abelian gauge theory:

$$\omega(x) \to \Lambda(x)\,\omega(x)\,\Lambda^{-1}(x) - d\Lambda(x)\,\Lambda^{-1}(x). \qquad (27)$$

The curvature forms $\Omega = (\Omega^\mu_\nu)$ are obtained from ω in exactly the same way as the Yang-Mills field strength from the gauge potential:

$$\Omega = d\omega + \omega \wedge \omega \qquad (28)$$

(second structure equation).
For a vector field *V*, with components V^α relative to $\{e_\alpha\}$, the covariant derivative *DV* is given by

$$DV^\alpha = dV^\alpha + \omega^\alpha_\beta V^\beta. \qquad (29)$$

Weyl generalizes this in a unique manner to spinor fields ψ belonging to representations ρ of *SL*(2,**C**):

$$D\psi = d\psi + \rho_*(\omega)\psi = d\psi + \frac{1}{4}\omega_{\alpha\beta}\sigma^{\alpha\beta}\psi. \qquad (30)$$

Here, ρ_* denotes the induced representation of the Lie algebra. For a Dirac field $\sigma^{\alpha\beta}$ are the familiar matrices

$$\sigma^{\alpha\beta} = \frac{1}{2}[\gamma^\alpha, \gamma^\beta]. \qquad (31)$$

(For 2-component Weyl fields one has similar expressions in terms of the Pauli matrices.)
With these tools the action principle for the coupled Einstein-Dirac system can be set up. In the massless case the Lagrangian is

$$\mathcal{L} = \frac{1}{16\pi G}R - i\bar{\psi}\gamma^\mu D_\mu\psi, \qquad (32)$$

where the first term is just the Einstein-Hilbert Lagrangian (which is linear in Ω). Weyl discusses, of course, immediately the consequences of the following two symmetries:
(i) local Lorentz invariance,
(ii) general coordinate invariance.

5.2 The New Form of the Gauge-Principle

All this is a kind of a preparation for the final section of Weyl's paper, which has the title "electric field". Weyl says:

> *We come now to the critical part of the theory. In my opinion the origin and necessity for the electromagnetic field is in the following. The components ψ_1 ψ_2 are, in fact, not uniquely determined by the tetrad but only to the extent that they can still be multiplied by an arbitrary "gauge-factor" $e^{i\lambda}$. The transformation of the ψ induced by a rotation of the tetrad is determined only up to such a factor. In special relativity one must regard this gauge-factor as a constant because here we have only a single point-independent tetrad. Not so in General Relativity; every point has its own tetrad and hence its own arbitrary gauge-factor; because by the removal of the rigid connection between tetrads at different points the gauge-factor necessarily becomes an arbitrary function of position.*

In this manner Weyl arrives at the gauge-principle in its modern form and emphasizes: "*From the arbitrariness of the gauge-factor in ψ appears the necessity of introducing the electromagnetic potential.*" The first term $d\psi$ in (30) has now to be replaced by the covariant gauge derivative $(d - iA)\psi$ and the nonintegrable scale factor (19) of the old theory is now replaced by a phase factor:

$$\exp\left(-\int_\gamma A\right) \to \exp\left(-i\int_\gamma A\right),$$

which corresponds to the replacement of the original gauge group $\mathbb{R}$ by the compact group $U(1)$. Accordingly, the original Gedankenexperiment of Einstein translates now to the Aharonov-Bohm effect, as was first pointed out by C. N. Yang in [22]. The close connection between gauge invariance and conservation of charge is again uncovered. The current conservation follows, as in the original theory, in two independent ways: On the one hand it is a consequence of the field equations for matter plus gauge invariance, at the same time, however, also of the field equations for the electromagnetic field plus gauge invariance. This corresponds to an identity in the coupled system of field equations which has to exist as a result of gauge invariance. All this is nowadays familiar to students of physics and does not need to be explained in more detail.

Much of Weyl's paper penetrated also into his classic book *The Theory of Groups and Quantum Mechanics* [23]. There he mentions also the transformation of his early gauge-theoretic ideas: "*This principle of gauge invariance is quite analogous to that previously set up by the author, on speculative grounds, in order to arrive at a unified theory of gravitation and electricity. But I now believe that this gauge invariance does not tie together electricity and gravitation, but rather electricity and matter.*"

When Pauli saw the full version of Weyl's paper he became more friendly and wrote [24]:

> *In contrast to the nasty things I said, the essential part of my last letter has since been overtaken, particularly by your paper in Z. f. Physik. For this reason I have afterward even regretted that I wrote to you. After studying your paper I believe that I have really understood what you wanted to do (this was not the case in respect of the little note in the Proc.Nat.Acad.). First let me emphasize that side of the matter concerning which I am in full agreement with you: your incorporation of spinor theory into gravitational theory. I am as dissatisfied as you are with distant parallelism and your proposal to let the tetrads rotate independently at different space-points is a true solution.*

In brackets Pauli adds:

> *Here I must admit your ability in Physics. Your earlier theory with $g'_{ik} = \lambda g_{ik}$ was pure mathematics and unphysical. Einstein was justified in criticizing and scolding. Now the hour of your revenge has arrived.*

Then he remarks in connection with the mass-problem:

> *Your method is valid even for the massive [Dirac] case. I thereby come to the other side of the matter, namely the unsolved difficulties of the Dirac theory (two signs of m_0) and the question of the 2-component theory. In my opinion these problems will not be solved by gravitation ...the gravitational effects will always be much too small.*

This remark indicates a major physical problem with classical spinor fields. Soon afterwards, beginning with Dirac's hole theory that led to the quantization of such fields with anticommutation relations, the problem was solved within special relativity, but remains in GR.

Many years later, Weyl summarized this early tortuous history of gauge theory in an instructive letter [25] to the Swiss writer and Einstein biographer C. Seelig, which we reproduce in the German original, followed by an English translation.

> *Aus dem Jahre 1918 datiert der von mir unternommene erste Versuch, eine einheitliche Feldtheorie von Gra-*

vitation und Elektromagnetismus zu entwickeln, und zwar auf Grund des Prinzips der Eichinvarianz, das ich neben dasjenige der Koordinaten-Invarianz stellte. Ich habe diese Theorie selber längst aufgegeben, nachdem ihr richtiger Kern: die Eichinvarianz, in die Quantentheorie herübergerettet ist als ein Prinzip, das nicht die Gravitation, sondern das Wellenfeld des Elektrons mit dem elektromagnetischen verknüpft. – Einstein war von Anfang dagegen, und das gab zu mancher Diskussion Anlass. Seinen konkreten Einwänden glaubte ich begegnen zu können. Schliesslich sagte er dann: "Na, Weyl, lassen wir das! So – das heisst auf so spekulative Weise, ohne ein leitendes, anschauliches physikalisches Prinzip – macht man keine Physik!" Heute haben wir in dieser Hinsicht unsere Standpunkte wohl vertauscht. Einstein glaubt, dass auf diesem Gebiet die Kluft zwischen Idee und Erfahrung so gross ist, dass nur der Weg der mathematischen Spekulation, deren Konsequenzen natürlich entwickelt und mit den Tatsachen konfrontiert werden müssen, Aussicht auf Erfolg hat, während mein Vertrauen in die reine Spekulation gesunken ist und mir ein engerer Anschluss an die quantenphysikalischen Erfahrungen geboten scheint, zumal es nach meiner Ansicht nicht genug ist, Gravitation und Elektromagnetismus zu einer Einheit zu verschmelzen. Die Wellenfelder des Elektrons und was es sonst noch an unreduzierbaren Elementarteilchen geben mag, müssen mit eingeschlossen werden.

The first attempt to develop a unified field theory of gravitation and electromagnetism dates to my first attempt in 1918, in which I added the principle of gauge-invariance to that of coordinate invariance. I myself have long since abandoned this theory in favour of its correct interpretation: gauge-invariance as a principle that connects electromagnetism not with gravitation but with the wave-field of the electron. — Einstein was against it [the original theory] from the beginning, and this led to many discussions. I thought that I could answer his concrete objections. In the end he said "Well, Weyl, let us leave it at that! In such a speculative manner, without any guiding physical principle, one cannot make Physics." Today one could say that in this respect we have exchanged our points of view. Einstein believes that in this field [Gravitation and Electromagnetism] the gap between ideas and experience is so wide that only the path of mathematical speculation, whose consequences must, of course, be developed and confronted with experiment, has a chance of success. Meanwhile my own confidence in pure speculation has diminished, and I see a need for a closer connection with quantum-physics experiments, since in my opinion it is not sufficient to unify Electromagnetism and Gravity. The wave-fields of the electron and whatever other irreducible elementary particles may appear must also be included.

Independently of Weyl, V. Fock [26] also incorporated the Dirac equation into General Relativity by using the same method. On the other hand, H. Tetrode [27], E. Schrödinger [28] and V. Bargmann [29] reached this goal by starting with space-time dependent γ-matrices, satisfying $\{\gamma_\mu, \gamma_\nu\} = 2g^{\mu\nu}$. A somewhat later work by L. Infeld and B. L. van der Waerden [30] is based on spinor analysis.

6 Gauge Invariance and QED

Gauge invariance became a serious problem when Heisenberg and Pauli began to work on a relativistically invariant Quantum Electrodynamics that eventually resulted in two important papers *On the Quantum Dynamics of Wave Fields* [32], [33]. Straightforward application of the canonical formalism led, already for the free electromagnetic field, to nonsensical results. Jordan and Pauli on the other hand, proceeded to show how to quantize the theory of the *free field* case by dealing only with the field strengths $F_{\mu\nu}(x)$. For these they found commutation relations at different space-time points in terms of the now famous invariant Jordan-Pauli distribution that are manifestly Lorentz invariant.

The difficulties concerned with applying the canonical formalism to the electromagnetic field continued to plague Heisenberg and Pauli for quite some time. By mid-1928 both were very pessimistic, and Heisenberg began to work on ferromagnetism [5]. In fall of 1928 Heisenberg discovered a way to bypass the difficulties. He added the term $-\frac{1}{2}\varepsilon(\partial_\mu A^\mu)^2$ to the Lagrangian, in which case the component π_0 of the canonical momenta

$$\pi_\mu = \frac{\partial L}{\partial(\partial_0 A_\mu)}$$

does no more vanish identically ($\pi_0 = -\varepsilon\partial_\mu A^\mu$). The standard canonical quantization scheme can then be applied. At the end of all calculations one could then take the limit $\epsilon \to 0$.

In their second paper, Heisenberg and Pauli stressed that the Lorentz condition cannot be imposed as an operator identity but only as a supplementary condition selecting admissible states. This discussion was strongly influenced by a paper of Fermi from May 1929.

For this and the further main developments during the early period of quantum field theory, we refer to chapter 1 of [34].

7 On Pauli's Invention of non-Abelian Kaluza- Klein Theory in 1953

There are documents which show that Wolfgang Pauli constructed in 1953 the first consistent generalization of the five-dimensional theory of Kaluza, Klein, Fock and others to a higher dimensional internal space. Because he saw no way to give masses to the gauge bosons, he refrained from publishing his results formally. This is still a largely unknown chapter of the early history of non-Abelian gauge and Kaluza-Klein theories.

Pauli described his detailed attempt of a non-Abelian generalization of Kaluza-Klein theories extensively in some letters to A. Pais, which have been published in Vol. IV, Part II of Pauli's collected letters [36], as well in two seminars in Zürich on November 16 and 23, 1953. The latter have later been written up in Italian by Pauli's pupil P. Gulmanelli [37]. An English translation of these notes by P. Minkowski is now

5 Pauli turned to literature. In a letter of 18 February 1929 he wrote from Zürich to Oskar Klein: "*For my proper amusement I then made a short sketch of a utopian novel which was supposed to have the title 'Gulivers journey to Urania' and was intended as a political satire in the style of Swift against present-day democracy. [...] Caught in such dreams, suddenly in January, news from Heisenberg reached me that he is able, with the aid of a trick ... to get rid of the formal difficulties that stood against the execution of our quantum electrodynamics.*" [6]

available on his home page. By specialization (independence of spinor fields on internal space) Pauli got all important formulae of Yang and Mills, as he later (Feb. 1954) pointed out in a letter to Yang [43], after a talk of Yang in Princeton. Pauli did not publish his study, because he was convinced that "*one will always obtain vector mesons with rest mass zero*" (Pauli to Pais, 6 Dec., 1953).

Figure 3: Wolfgang Pauli around 1956.

7.1 The Pauli letters to Pais

At the Lorentz-Kammerlingh Onnes conference in Leiden (22-27 June 1953) A. Pais talked about an attempt of describing nuclear forces based on isospin symmetry and baryon number conservation. In this contribution he introduced fields, which do not only depend on the spacetime coordinates *x*, but also on the coordinates ω of an internal isospin space. The isospin group acted, however, globally, i.e., in a spacetime-independent manner.
During the discussion following the talk by Pais, Pauli said:

> *"...I would like to ask in this connection whether the transformation group with constant phases can be amplified in a way analogous to the gauge group for electromagnetic potentials in such a way that the meson-nucleon interaction is connected with the amplified group..."*

Stimulated by this discussion, Pauli worked on the problem, and wrote on July 25, 1953 a long technical letter to Pais [38], with the motto: "Ad *usum Delfini* only". This letter begins with a personal part in which Pauli says that "*the whole note for you is of course written in order to drive you further into the real virgin-country*". The note has the interesting title:

> *Written down July 22-25 1953, in order to see how it looks. Meson-Nucleon Interaction and Differential Geometry."*

In this manuscript, Pauli generalizes the original Kaluza-Klein theory to a six-dimensional space and arrives through dimensional reduction at the essentials of an $SU(2)$ gauge theory. The extra-dimensions form a two-sphere S^2 with spacetime dependent metrics on which the $SU(2)$ operates in a space-time-dependent manner. Pauli emphasizes that this transformation group "*seems to me therefore the **natural generalization of the gauge-group** in case of a two-dimensional spherical surface*". He then develops in 'local language' the geometry of what we now call a fibre bundle with a homogeneous space as typical fiber (in this case $SU(2)/U(1)$).

Since it is somewhat difficult to understand exactly what Pauli did, we give some details, using more familiar formulations and notations [4].

Pauli considers the six-dimensional total space $M \times S^2$, where S^2 is the two-sphere on which $SO(3)$ acts in the canonical manner. He distinguishes among the diffeomorphisms (coordinate transformations) those which leave the space-time manifold M pointwise fixed and induce space-time-dependent rotations on S^2:

$$(x, y) \to [x, R(x) \cdot y]. \qquad (33)$$

Then Pauli postulates a metric on $M \times S^2$ that is supposed to satisfy three assumptions. These led him to what is now called the non-Abelian Kaluza-Klein ansatz: The metric $\hat{g}$ on the total space is constructed from a space-time metric g, the standard metric γ on S^2, and a Lie-algebra-valued 1-form,

$$A = A^a T_a \;,\; A^a = A^a_\mu dx^\mu, \qquad (34)$$

on M (T_a, $a = 1, 2, 3$, are the standard generators of the Lie algebra of $SO(3)$) as follows: If $K^i_a \partial/\partial y^i$ are the three Killing fields on S^2, then

$$\hat{g} = g - \gamma_{ij}[dy^i + K^i_a(y)A^a] \otimes [dy^j + K^j_a(y)A^a]. \qquad (35)$$

In particular, the non-diagonal metric components are

$$\hat{g}_{\mu i} = A^a_\mu(x)\gamma_{ij}K^j_a. \qquad (36)$$

Pauli does not say that the coefficients of A^a_μ in Eq. (36) are the components of the three independent Killing fields. This is, however, his result, which he formulates in terms of homogeneous coordinates for S^2. He determines the transformation behavior of A^a_μ under the group (1) and finds in matrix notation what he calls "*the generalization of the gauge group*":

$$A_\mu \to R^{-1} A_\mu R + R^{-1} \partial_\mu R. \qquad (37)$$

With the help of A_μ, he defines a covariant derivative, which is used to derive "*field strengths*" by applying a generalized curl to A_μ. This is exactly the field strength that was later introduced by Yang and Mills. To our knowledge, apart from Klein's 1938 paper, it appears here for the first time. Pauli says that "*this is the **true** physical field, the analog of the **field strength***" and he formulates what he considers to be his "*main result*":

> *The vanishing of the field strength is necessary and sufficient for the $A^a_\mu(x)$ in the whole space to be transformable to zero.*

It is somewhat astonishing that Pauli did not work out the Ricci scalar for $\hat{g}$ as for the Kaluza-Klein theory. One reason may be connected with his remark on the Kaluza-Klein theory in Note 23 of his relativity article [39] concerning the five dimensional curvature scalar (p. 230):

> *There is, however, no justification for the particular choice of the five-dimensional curvature scalar P as integrand of the action integral, from the standpoint of the restricted group of the cylindrical metric (gauge group). The open problem of finding such a justification seems to point to an amplification of the transformation group.*

In a second letter [40], Pauli also studies the dimensionally reduced Dirac equation and arrives at a mass operator that is closely related to the Dirac operator in internal space (S^2, γ). The eigenvalues of the latter operator had been determined by him long before [41]. Pauli concludes with the statement: "*So this leads to some rather unphysical **shadow particles**"*.

Pauli's main concern was that the gauge bosons had to be massless, as in quantum electrodynamics. He emphasized this mass problem repeatedly, most explicitly in the second letter [40] to Pais on December 6, 1953, after he had made some new calculations and had given the two seminar lectures in Zürich already mentioned. He adds to the Lagrangian what we now call the Yang-Mills term for the field strengths and says that "*one will always obtain vector mesons with rest-mass zero (and the rest-mass if at all finite, will always remain zero by all interactions with nucleons permitting the gauge group).*" To this Pauli adds: "*One could try to find other meson fields*", and he mentions, in particular, the scalar fields which appear in the dimensional reduction of the higher-dimensional metric. In view of the Higgs mechanism this is an interesting remark.

Pauli learned about the related work of Yang and Mills in late February, 1954, during a stay in Princeton, when Yang was invited by Oppenheimer to return to Princeton and give a seminar on his joint work with Mills. About this seminar Yang reports [42]: "*Soon after my seminar began, when I had written down on the blackboard* $(\partial_\mu - i\epsilon B_\mu)\Psi$, *Pauli asked:* '*What is the mass of this field* B_μ*?*', *I said we did not know. Then I resumed my presentation, but soon Pauli asked the same question again. I said something to the effect that that was a very complicated problem, we had worked on it and had come to no conclusion. I still remember his repartee: 'That is no sufficient excuse.' I was so taken aback that I decided, after a few moments' hesitation to sit down. There was general embarrassment. Finally Oppenheimer said, 'we should let Frank proceed.' Then I resumed and Pauli did not ask any more questions during the seminar.*" (For more on this encounter, see [42].)

In a letter to Yang [43] shortly after Yang's Princeton seminar, Pauli repeats:

> *But I was and still am disgusted and discouraged of the vector field corresponding to particles with zero rest-mass (I do not take your excuses for it with 'complications' seriously) and the difficulty with the group due to the distinction of the electromagnetic field remains.*

Formally, Pauli had, however, all important equations, as he shows in detail, and he concludes the letter with the sentence: "On the other hand you see, that your equations can easily be generalized to include the ω- space" (the internal space). As already mentioned, the technical details have been written up by Pauli's pupil P. Gulmanelli [37] and have recently been translated by P. Minkowski from Italian to English.

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